The Newton's Method

If $f(x)$, $f'(x)$, and $f'(x)$ are continuous near a root p, then this extra information regarding the nature of $f(x)$ can be used to develop algorithms that will produce sequences $\{p_k\}$ that converge faster to p than either the bisection or false position method. The Newton-Raphson (or simply Newton's) method is one of the most useful and best known algorithms that relies on the continuity of $f'(x)$ and $f'(x)$. The method is attributed to Sir Isac Newton (1643-1727) and Joseph Ralphson (1648-1715).

Theorem (Newton-Ralphson Theorem). Assume that $f \in \mathbf{C}^2[a, b]$ and there exists a number $p \in [a, b]$, where $f(p) = 0$. If $f'(p) \neq 0$, then there exists a $\delta > 0$ such that the sequence $\{p_k\}_{k=0}^{\infty}$ defined by the iteration

$$p_{k+1} = g(p_k) = p_k - \frac{f(p_k)}{f'(p_k)} \quad \text{for} \quad k = 0, 1, \ldots$$

will converge to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$.

Algorithm (Newton-Raphson Iteration). To find a root of $f(x) = 0$ given an initial approximation p_0 using the iteration

$$p_{k+1} = p_k - \frac{f(p_k)}{f'(p_k)} \quad \text{for} \quad k = 0, 1, \cdots, m.$$

Mathematica Subroutine (Newton-Raphson Iteration).

```
NewtonRaphson[x0_, max_] :=
 Module[{},
  k = 0;
  p0 = N[x0];
  Print["p₀ = ", PaddedForm[p0, {16, 16}], ",    f[p₀] = ",
   NumberForm[f[p0], 16] ];
  p1 = p0;
  While[ k < max,
   p0 = p1;
   p1 = p0 - f[p0]/f'[p0];
   k = k + 1;
   Print["p"_k, " = ", PaddedForm[p1, {16, 16}], ",    f[", "p"_k,
    "] = ", NumberForm[f[p1], 16] ]; ];
  Print["  p  = ", NumberForm[p1, 16] ];
  Print[" Δp  = ±", Abs[p1 - p0] ];
  Print["f[p] = ", NumberForm[f[p1], 16] ]; ]
```

Example 1. Use Newton's method to find the three roots of the cubic polynomial $f[x] = 4x^3 - 15x^2 + 17x - 6$.

Determine the Newton-Raphson iteration formula $g[x] = x - \frac{f[x]}{f'[x]}$ that is used. Show details of the computations for the starting value $p_0 = 3$.

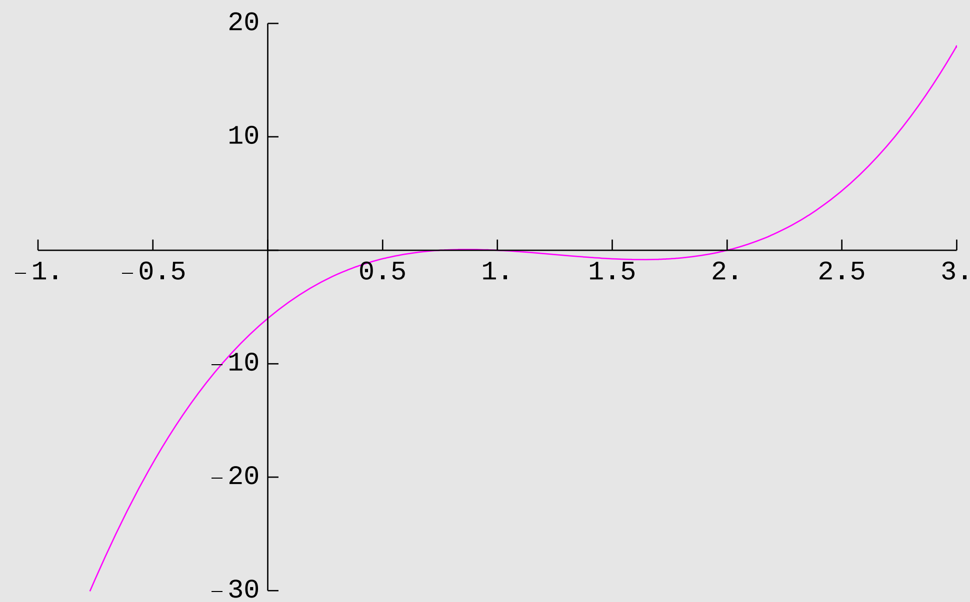
Solution

```
f[x_] = 4 x^3 - 15 x^2 + 17 x - 6;
Print["f[x] = ", f[x] ];
```

$f[x] = -6 + 17x - 15x^2 + 4x^3$

Graph the function.

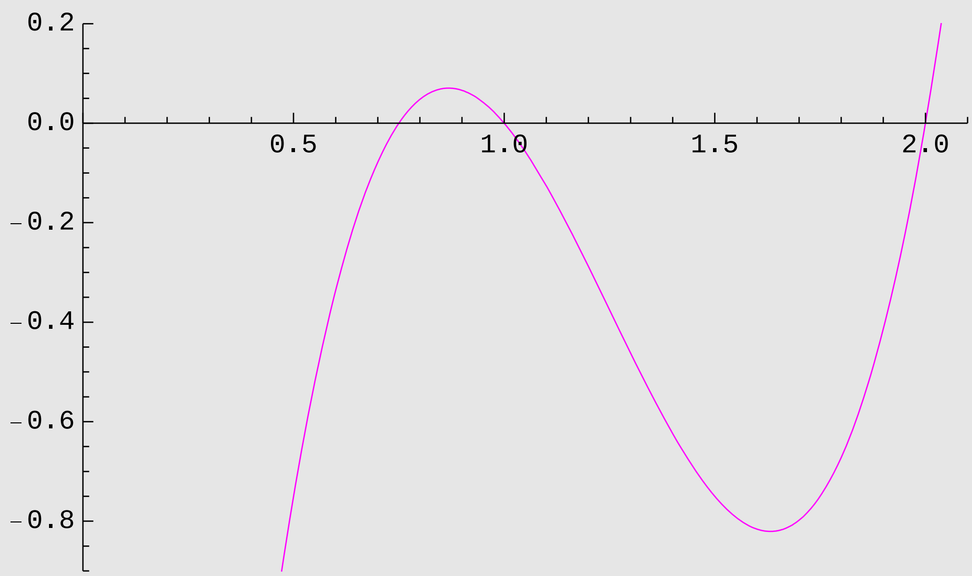
```
Needs["Graphics`Colors`"]
Plot[f[x], {x, -1, 3}, PlotRange → {{-1, 3}, {-30, 20}},
 Ticks → {Range[-1, 3, 0.5`], Range[-30, 20, 10]}, PlotStyle → Magenta]
Print["f[x] = ", f[x]];
```



f[x] = $-6+17x-15x^2+4x^3$

How many real roots are there ? Really !

```
Plot[f[x], {x, 0, 3}, PlotRange → {{0, 2.1`}, {-0.9`, 0.2`}},
 PlotStyle → Magenta]
Print["f[x] = ", f[x]];
```



f[x] = $-6+17x-15x^2+4x^3$

The Newton-Raphson iteration formula g[x] is

```
g[x_] = x - f[x]/f'[x];
Print["g[x] = ", g[x] ];
g[x_] = Simplify[ g[x] ];
Print["g[x] = ", g[x] ];
```

g[x] = $x-\frac{-6+17x-15x^2+4x^3}{17-30x+12x^2}$

g[x] = $\frac{6-15x^2+8x^3}{17-30x+12x^2}$

Starting with $p_0 = 3$, Use the Newton-Raphson method to find a numerical approximation to the root. First, do the iteration one step at a time. Type each of the following commands in a separate cell and execute them one at a time.

```
p0 = 3.0
```

```
3.
```

```
p1 = g[p0]
```

```
2.48571
```

```
p2 = g[p1]
```

```
2.18342
```

```
p3 = g[p2]
```

```
2.04045
```

```
p4 = g[p3]
```

```
2.00265
```

```
p5 = g[p4]
```

```
2.00001
```

```
p6 = g[p5]
```

```
2.
```

```
NewtonRaphson[3.0, 7];
```

p_0 = 3.0000000000000000, $f[p_0]$ = 18.

p_1 = 2.4857142857142860, $f[p_1]$ = 5.010192419825074

p_2 = 2.1834197620337600, $f[p_2]$ = 1.244567116269891

p_3 = 2.0404526629830990, $f[p_3]$ = 0.2172558662514135

p_4 = 2.0026544732145300, $f[p_4]$ = 0.01333585694116124

p_5 = 2.0000125925878950, $f[p_5]$ = 0.00006296436663433269

p_6 = 2.0000000002854240, $f[p_6]$ = $1.427117979346804 \times 10^{-9}$

p_7 = 2.0000000000000000, $f[p_7]$ = 0.

p = 2.

$\triangle p$ = $\pm 2.85424 \times 10^{-10}$

f[p] = 0.

From the second graph we see that there are two other real roots, use the starting values 0.0 and 1.4 to find them.

First, use the starting value $p_0 = 0.0$.

```
NewtonRaphson[0.0, 8];
```

p_0 = 0.0000000000000000, $f[p_0]$ = −6.

p_1 = 0.3529411764705882, $f[p_1]$ = −1.692652147364137

p_2 = 0.5670227828549363, $f[p_2]$ = −0.4541102868356983

p_3 = 0.6850503150510711, $f[p_3]$ = −0.1075938266370038

p_4 = 0.7367776746893979, $f[p_4]$ = −0.01758613258850872

p_5 = 0.7492433382396959, $f[p_5]$ = −0.00094926415536456

p_6 = 0.7499972689032873, $f[p_6]$ = $-3.4139156444013 \times 10^{-6}$

p_7 = 0.7499999999641983, $f[p_7]$ = $-4.475175785501051 \times 10^{-11}$

p_8 = 0.7499999999999996, $f[p_8]$ = 0.

p = 0.7499999999999996

$\triangle p$ = $\pm 3.58014 \times 10^{-11}$

f[p] = 0.

Then use the starting value $p_0 = 1.4$.

```
NewtonRaphson[1.4, 5];
```

p_0 = 1.4000000000000000, f[p_0] = −0.6240000000000023

p_1 = 0.9783783783783780, f[p_1] = 0.02017870609835803

p_2 = 1.0017155262247030, f[p_2] = −0.001724335120005804

p_3 = 1.0000086994622170, f[p_3] = $-8.69968925343301 \times 10^{-6}$

p_4 = 1.0000000002270280, f[p_4] = $-2.270255095027096 \times 10^{-10}$

p_5 = 1.0000000000000020, f[p_5] = 0.

p = 1.000000000000002

Δp = $\pm 2.27026 \times 10^{-10}$

f[p] = 0.

Compare our result with *Mathematica*'s built in numerical root finder.

```
solset = NSolve[f[x] == 0, x];
NumberForm[TableForm[solset], 11]
```

```
x → 0.75
x → 1.
x → 2.
```

This can also be done with *Mathematica*'s built in symbolic solve procedure.

```
solset = Solve[f[x] == 0, x];
Print["f[x] = ", Factor[ f[x] ] ];
NumberForm[TableForm[solset], 11]
```

f[x] = (-2 + x) (-1 + x) (-3 + 4 x)

$x \to \frac{3}{4}$
$x \to 1$
$x \to 2$

Definition (Order of a Root) Assume that $f(x)$ and its derivatives $f'(x)$, ..., $f^{(m)}(x)$ are defined and continuous on an interval about $x = p$. We say that $f(x) = 0$ has a root of order m at $x = p$ if and only if

$$f(p) = 0,\ f'(p) = 0,\ f''(p) = 0,\ \ldots,\ f^{(m-1)}(p) = 0,\ f^{(m)}(p) \neq 0.$$

A root of order $m = 1$ is often called a **simple root**, and if $m > 1$ it is called a **multiple root**. A root of order $m = 2$ is sometimes called a **double root**, and so on. The next result will illuminate these concepts.

Definition (Order of Convergence) Assume that p_n converges to p, and set $E_n = p - p_n$ for $n \geq 0$. If two positive constants $A \neq 0$ and $R > 0$ exist, and

$$\lim_{n \to \infty} \frac{|p - p_{n+1}|}{|p - p_n|^R} = \lim_{n \to \infty} \frac{|E_{n+1}|}{|E_n|^R} = A,$$

then the sequence is said to converge to p with **order of convergence R**. The number A is called the **asymptotic error constant**. The cases $R = 1, 2$ are given special consideration.

(i) If $R = 1$, the convergence of $\{p_k\}_{k=0}^{\infty}$ is called **linear**.

(ii) If $R = 2$, the convergence of $\{p_k\}_{k=0}^{\infty}$ is called **quadratic**.

Theorem (Convergence Rate for Newton-Raphson Iteration)
Assume that Newton-Raphson iteration produces a sequence $\{p_k\}_{k=0}^{\infty}$ that converges to the root p of the function $f(x)$.

If p is a simple root, then convergence is quadratic and
$\left|E_{k+1}\right| \approx \frac{|f''(p)|}{2|f'(p)|}(|E_k|)^2$ for k sufficiently large.

If p is a multiple root of order m, then convergence is linear and
$\left|E_{k+1}\right| \approx \frac{m-1}{m}\left|E_k\right|$ for k sufficiently large.

Example 2. Use Newton's method to find the roots of the cubic polynomial $f[x] = x^3 - 3x + 2$.

2 (a) Fast Convergence. Investigate quadratic convergence at the simple root $p = -2$, using the starting value $p_0 = -2.4$

2 (b) Slow Convergence. Investigate linear convergence at the double root $p = 1$, using the starting value $p_0 = 1.2$

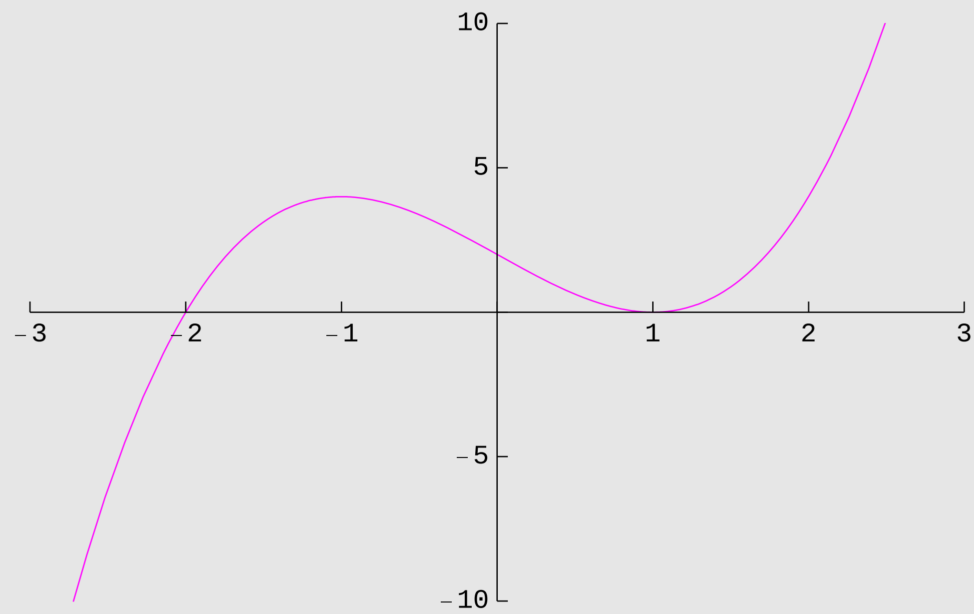
Solution

```
f[x_] = x^3 - 3 x + 2;
Print["f[x] = ", f[x]];
```

$f[x] = 2 - 3x + x^3$

Graph the function.

```
Needs["Graphics`Colors`"]
Plot[f[x], {x, -3, 3}, PlotRange → {{-3, 3}, {-10, 10}},
 Ticks → {Range[-3, 3, 1], Range[-10, 10, 5]}, PlotStyle → Magenta]
Print["f[x] = ", f[x]];
```



f[x] = $2 - 3x + x^3$

The Newton-Raphson iteration formula g[x] is

```
g[x_] = x - f[x]/f'[x];
Print["g[x] = ", g[x] ];
g[x_] = Simplify[ g[x] ];
Print["g[x] = ", g[x] ];
```

$$g[x] = x - \frac{2 - 3x + x^3}{-3 + 3x^2}$$

$$g[x] = \frac{2\left(1 + x + x^2\right)}{3\,(1 + x)}$$

2 (a) Fast Convergence. Investigate quadratic convergence at the simple root $p = -2$, using the starting value $p_0 = -2.4$

First, do the iteration one step at a time.

Type each of the following commands in a separate cell and execute them one at a time.

```
p0 = -2.4
```

```
-2.4
```

```
p1 = g[p0]
```

```
-2.07619
```

```
p2 = g[p1]
```

```
-2.0036
```

```
p3 = g[p2]
```

```
-2.00001
```

```
p4 = g[p3]
```

```
-2.
```

```
p5 = g[p4]
```

```
-2.
```

```
p6 = g[p5]
```

```
-2.
```

Notice that convergence is fast and the sequence is converging to the simple root $p = -2$

```
NewtonRaphson[-2.4, 7];
```

$p_0 = -2.4000000000000000,\quad f[p_0] = -4.623999999999999$

$p_1 = -2.0761904761904760,\quad f[p_1] = -0.7209865025375244$

$p_2 = -2.0035960106756570,\quad f[p_2] = -0.03244173033865483$

$p_3 = -2.0000085899722210,\quad f[p_3] = -0.00007731019271695061$

$p_4 = -2.0000000000491910,\quad f[p_4] = -4.427214150837244 \times 10^{-10}$

$p_5 = -2.0000000000000000,\quad f[p_5] = 0.$

$p_6 = -2.0000000000000000,\quad f[p_6] = 0.$

$p_7 = -2.0000000000000000,\quad f[p_7] = 0.$

$p = -2.$

$\triangle p = \pm 0.$

$f[p] = 0.$

At the simple root $p = -2$ we can explore the relationship $\left| E_{k+1} \right| \approx \frac{|f''(p)|}{2|f'(p)|} (\,| E_k |)^2$ for k sufficiently large.

This will be done by investigating the ratio $\frac{|E_{k+1}|}{(|E_k|)^2} \approx \frac{|f''(p)|}{2|f'(p)|}$ for k sufficiently large.

```
Nk = 4;
Pk = NestList[g, -2.4, Nk];
Vk = Table[i - 1, {i, 1, Nk + 1}];
Ek = -2 - Pk;
Qk = Take[Abs[Ek], -Nk] / Take[Ek^2, Nk];
Qk = Append[Qk, " "];
Tvals = Transpose[{Vk, Pk, Ek, Qk}];
NumberForm[
 TableForm[Tvals,
  TableHeadings ->
   {None, {"k", "pk", "Ek=p-pk", "|SubscriptBox[E, k + 1]| / (|SubscriptBox[E, k]|)^2"}},
  TableSpacing -> {3, 1}], 15]
```

k	p_k	$E_k = p - p_k$	$\frac{\lvert \text{SubscriptBox}[E,k+1] \rvert}{(\lvert \text{SubscriptBox}[E,k] \rvert)^2}$
0	-2.4	0.4	0.476190476190475
1	-2.07619047619048	0.0761904761904759	0.619469026548608
2	-2.00359601067566	0.00359601067565629	0.664277916183774
3	-2.00000858997222	$8.58997222108471 \times 10^{-6}$	0.666654809538057
4	-2.00000000004919	$4.91908735966717 \times 10^{-11}$	

Evaluate the quantity $\frac{|f''(p)|}{2|f'(p)|}$ at the root $p = -2$.

```
Abs[f''[-2]]      2
-------------  == -  == 0.6666666666666666
2 Abs[f'[-2]]     3
```

```
True
```

Which is close to the computed value $\frac{|E_3|}{(|E_3|)^2}$ = 0.666654809469126

2 (b) Slow Convergence. Investigate linear convergence at the double root $p = 1$, using the starting value $p_0 = 1.2$

First, do the iteration one step at a time.

Type each of the following commands in a separate cell and execute them one at a time.

```
p0 = 1.2
```

```
1.2
```

```
p1 = g[p0]
```

```
1.10303
```

```
p2 = g[p1]
```

```
1.05236
```

```
p3 = g[p2]
```

```
1.0264
```

```
p4 = g[p3]
```

```
1.01326
```

```
p5 = g[p4]
```

```
1.00664
```

```
p6 = g[p5]
```

```
1.00333
```

Notice that convergence is slow, but the sequence is converging to the double root $p = 1$

```
NewtonRaphson[1.2, 25];
```

p_0 = 1.2000000000000000, $f[p_0]$ = 0.1280000000000001

p_1 = 1.1030303030303030, $f[p_1]$ = 0.03293942176586806

p_2 = 1.0523564171979160, $f[p_2]$ = 0.00836710238417515

p_3 = 1.0264008140553680, $f[p_3]$ = 0.002109410394502964

p_4 = 1.0132577338719060, $f[p_4]$ = 0.0005296328010917506

p_5 = 1.0066434177726740, $f[p_5]$ = 0.0001326982063480919

p_6 = 1.0033253746264610, $f[p_6]$ = 0.00003321112159881956

p_7 = 1.0016636072932840, $f[p_7]$ = $8.30737186041652 \times 10^{-6}$

p_8 = 1.0008320340873970, $f[p_8]$ = $2.077418169044165 \times 10^{-6}$

p_9 = 1.0004160747097450, $f[p_9]$ = $5.194265224606198 \times 10^{-7}$

p_{10} = 1.0002080517783450, $f[p_{10}]$ = $1.298656329140613 \times 10^{-7}$

p_{11} = 1.0001040294960360, $f[p_{11}]$ = $3.246753399466229 \times 10^{-8}$

p_{12} = 1.0000520156497570, $f[p_{12}]$ = $8.1170243859674 \times 10^{-9}$

p_{13} = 1.0000260080497260, $f[p_{13}]$ = $2.029273638015638 \times 10^{-9}$

p_{14} = 1.0000130040806280, $f[p_{14}]$ = $5.073206299499589 \times 10^{-10}$

p_{15} = 1.0000065020532280, $f[p_{15}]$ = $1.268303240209434 \times 10^{-10}$

p_{16} = 1.0000032510311470, $f[p_{16}]$ = $3.170774753868955 \times 10^{-11}$

p_{17} = 1.0000016255111940, $f[p_{17}]$ = $7.926992395823618 \times 10^{-12}$

p_{18} = 1.0000008127426750, $f[p_{18}]$ = $1.981748098955904 \times 10^{-12}$

p_{19} = 1.0000004063517890, $f[p_{19}]$ = $4.953815135877448 \times 10^{-13}$

p_{20} = 1.0000002031692980, $f[p_{20}]$ = $1.239008895481675 \times 10^{-13}$

p_{21} = 1.0000001015292060, $f[p_{21}]$ = $3.064215547965432 \times 10^{-14}$

p_{22} = 1.0000000512281560, $f[p_{22}]$ = $7.993605777301127 \times 10^{-15}$

p_{23} = 1.0000000252216060, $f[p_{23}]$ = $1.998401444325282 \times 10^{-15}$

p_{24} = 1.0000000120159870, $f[p_{24}]$ = $6.661338147750939 \times 10^{-16}$

p_{25} = 1.0000000027764380, $f[p_{25}]$ = 0.

p = 1.000000002776438

$\triangle p$ = $\pm 9.23955 \times 10^{-9}$

$f[p]$ = 0.

At the double root $p = 1$ we can explore the relationship $\left|E_{k+1}\right| \approx \frac{1}{2}\left|E_k\right|$ for k sufficiently large.

This will be done by investigating the ratio $\frac{|E_{k+1}|}{|E_k|} \approx \frac{1}{2}$ for k sufficiently large.

```
Nk = 24;
Pk = NestList[g, 1.2, Nk];
Vk = Table[i - 1, {i, 1, Nk + 1}];
Ek = 1 - Pk;
Qk = Take[Abs[Ek], -Nk] / Take[Abs[Ek], Nk];
Qk = Append[Qk, " "];
Tvals = Transpose[{Vk, Pk, Ek, Qk}];
NumberForm[
 TableForm[Tvals,
  TableHeadings ->
   {None, {"k", "pk", "Ek=p-pk", "|SubscriptBox[E, k + 1]| / |SubscriptBox[E, k]|"}},
  TableSpacing -> {3, 1}], 15]
```

k	p_k	$E_k = p - p_k$	$\frac{\lvert\text{SubscriptBox}[E, k+1]\rvert}{\lvert\text{SubscriptBox}[E, k]\rvert}$
0	1.2	−0.2	0.515151515151514
1	1.1030303030303	−0.103030303030303	0.508165225744476
2	1.05235641719792	−0.0523564171979156	0.504251732038289
3	1.02640081405537	−0.0264008140553682	0.502171404415834
4	1.01325773387191	−0.0132577338719055	0.501097535737622
5	1.00664341777268	−0.00664341777267707	0.500551785277615
6	1.00332537462646	−0.00332537462645854	0.500276654562143
7	1.00166360729329	−0.00166360729329051	0.500138518720712
8	1.0008320340874	−0.000832034087399292	0.500069307340506
9	1.00041607470977	−0.000416074709769454	0.500034665680735
10	1.0002080517784	−0.000208051778397778	0.500017335844799
11	1.00010402949595	−0.000104029495952229	0.500008668671489
12	1.00005201564977	−0.0000520156497736402	0.500004334522064
13	1.00002600805035	−0.0000260080503498017	0.500002167312679
14	1.00001300408154	−0.0000130040815424781	0.500001083672986
15	1.00000650205486	$-6.50205486341093\times10^{-6}$	0.500000541822512
16	1.00000325103096	$-3.25103095466517\times10^{-6}$	0.500000270876813
17	1.00000162551636	$-1.62551635796149\times10^{-6}$	0.500000135506633
18	1.0000008127584	$-8.12758399248992\times10^{-7}$	0.500000067753298
19	1.00000040637926	$-4.06379254691558\times10^{-7}$	0.500000033876644
20	1.00000020318964	$-2.03189641112545\times10^{-7}$	0.500000016391924
21	1.00000010159482	$-1.01594823886941\times10^{-7}$	0.500000005463974
22	1.00000005079741	$-5.07974124985822\times10^{-8}$	0.500000004371179
23	1.00000002539871	$-2.53987064713357\times10^{-8}$	0.500000004371179
24	1.00000001269935	$-1.26993533466902\times10^{-8}$	

Compare our result with *Mathematica*'s built in numerical root finder.

```
solset = NSolve[f[x] == 0, x];
NumberForm[TableForm[solset], 11]
```

```
x → -2.
x → 1.
x → 1.
```

This can also be done with *Mathematica*'s built in symbolic solve procedure.

```
solset = Solve[f[x] == 0, x];
Print["f[x] = ", Factor[ f[x] ] ];
NumberForm[TableForm[solset], 11]
```

```
f[x] = (-1 + x)^2 (2 + x)
```

```
x → -2
x → 1
x → 1
```

Reduce the volume of printout.

After you have debugged you program and it is working properly, delete the unnecessary print statements

```
Print["p0 = ", PaddedForm[N[p0], {11, 11}], ",    f[p0] = ", f[p0] ];
Print["p", k, " = ", PaddedForm[N[p1], {11, 11}], ",    f[p", k,
  "] = ", f[p1] ];
```

```
p0 =  1.00000001200,    f[p0] = 6.66134×10^-16
p25 =  1.00000000280,    f[p25] = 0.
```

Concise Program for the Newton-Raphson Method

```
NewtonRaphson[x0_, max_] :=
  Module[{},
   k = 0;
   p0 = N[x0];
   p1 = p0;
   While[ k < max,
    p0 = p1;
    p1 = p0 - f[p0]/f'[p0];
    k = k + 1; ];
   Print["  p   = ", NumberForm[p1, 16] ];
   Print[" Δp   = ±", Abs[p1 - p0] ];
   Print["f[p]  = ", NumberForm[f[p1], 16] ]; ];
```

Now test this subroutine using the function in Example 1.

```
f[x_] = 4 x^3 - 15 x^2 + 17 x - 6;
Print["f[x] = ", f[x] ];
```

$f[x] = -6 + 17 x - 15 x^2 + 4 x^3$

```
NewtonRaphson[3.0, 7];
```

```
  p   = 2.
 Δp   = ±2.85424×10^-10
f[p]  = 0.
```

```
NewtonRaphson[0.0, 8];
```

```
  p   = 0.7499999999999996
 Δp   = ±3.58014×10^-11
f[p]  = 0.
```

```
NewtonRaphson[1.4, 5];
```

```
  p   = 1.000000000000002
 Δp   = ±2.27026×10^-10
f[p]  = 0.
```

Error Checking in the Newton-Raphson Method

Error checking can be added to the Newton-Raphson method. Here we have added a third parameter δ to the subroutine which estimate the accuracy of the numerical solution.

```
NewtonRaphson[x0_, δ_, max_] :=
  Module[{},
   k = 0;
   p0 = N[x0];
   Δp = 1;
   While[ And[ k < max, δ < Abs[Δp] ],
    Δp = f[p0]/f'[p0];
    p1 = p0 - Δp;
    k = k + 1;
    p0 = p1; ];
   Print["  p   = ", NumberForm[p1, 11] ];
   Print[" Δp   = ±", Abs[Δp] ];
   Print["f[p]  = ", NumberForm[f[p1], 11] ]; ];
```

The following subroutine call uses a maximum of 20 iterations, just to make sure enough iterations are performed. However, it will terminate when the difference between consecutive iterations is less than 10^{-10}. By interrogating k afterward we can see how many iterations were actually performed.

```
f[x_] = 4 x^3 - 15 x^2 + 17 x - 6;
Print["f[x] = ", f[x] ];
```

f[x] = $-6 + 17 x - 15 x^2 + 4 x^3$

```
NewtonRaphson[0.0, 10^-10, 20];
```

$p = 0.75$

$\Delta p = \pm 3.58014 \times 10^{-11}$

$f[p] = 0.$

Various Scenarios for Newton-Raphson Iteration.

```
NewtonRaphson[x0_, max_] :=
 Module[{},
  k = 0;
  p0 = N[x0];
  Print["f[x] = ", f[x] ];
  g[x_] = x - f[x]/f'[x];
  Print["g[x] = x - f[x]/f'[x]"];
  Print["g[x] = ", g[x] ];
  Print["g[x] = ", Simplify[g[x]]];
  Print["  p0 = ", PaddedForm[p0, {16, 16}], ",    f[p0] = ",
   NumberForm[f[p0], 16] ];
  p1 = p0;
  While[ k < max,
   p0 = p1;
   p1 = p0 - f[p0]/f'[p0];
   k = k + 1;
   Print["  p"k, " = ", PaddedForm[p1, {16, 16}], ",    f[", "p"k,
    "] = ", NumberForm[f[p1], 16] ]; ];
  Print[""];
  Print["f[x] = ", f[x] ];
  Print["  p  = ", NumberForm[p1, 16] ];
  Print[" Δp  = ±", Abs[p1 - p0] ];
  Print["f[p] = ", NumberForm[f[p1], 16] ]; ]
```

Example 3. Fast Convergence Find the solution to $3 \operatorname{Exp}[x] - 4 \operatorname{Cos}[x] = 0$. Use the starting approximation $p_0 = 1.0$.

Solution

```
f[x_] = 3 ⅇ^x - 4 Cos[x];
(p0 = 1.`;) (n = 6;) (NewtonRaphson[p0, n];)
```

$f[x] = 3\,e^{x} - 4\,\mathrm{Cos}[x]$

$$g[x] = x - \frac{f[x]}{f'[x]}$$

$$g[x] = x - \frac{3\,e^{x} - 4\,\mathrm{Cos}[x]}{3\,e^{x} + 4\,\mathrm{Sin}[x]}$$

$$g[x] = x + \frac{-3\,e^{x} + 4\,\mathrm{Cos}[x]}{3\,e^{x} + 4\,\mathrm{Sin}[x]}$$

$p_0 = 1.0000000000000000$, $f[p_0] = 5.993636261904577$

$p_1 = 0.4797520156057185$, $f[p_1] = 1.298583433809675$

$p_2 = 0.2857383591311282$, $f[p_2] = 0.1544175142133715$

$p_3 = 0.2555769004716556$, $f[p_3] = 0.003548454019948188$

$p_4 = 0.2548504777343278$, $f[p_4] = 2.042950823177847 \times 10^{-6}$

$p_5 = 0.2548500590289893$, $f[p_5] = 6.785683126508957 \times 10^{-13}$

$p_6 = 0.2548500590288503$, $f[p_6] = 0.$

$f[x] = 3\,e^{x} - 4\,\mathrm{Cos}[x]$

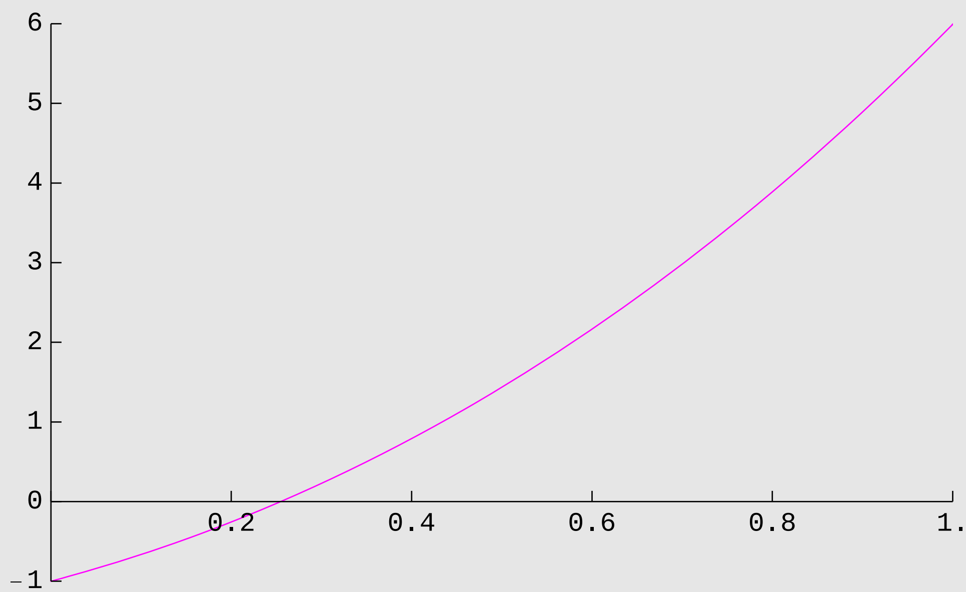
$p = 0.2548500590288503$

$\triangle p = \pm 1.39055 \times 10^{-13}$

$f[p] = 0.$

Null^{3}

```
Needs["Graphics`Colors`"]
Plot[f[x], {x, 0.`, 1.`}, PlotRange → {{0, 1.`}, {-1, 6}},
 Ticks → {Range[0, 1, 0.2`], Range[-1, 6, 1]}, PlotStyle → Magenta]
```



Example 4. Slow Convergence Find the solution to $1 - 10x + 25x^2 = 0$. Use the starting approximation $p_0 = 1.0$.

Solution

```
(f[x_] = 1 - 10 x + 25 x^2;) (p0 = 1.`;) (n = 25;) (NewtonRaphson[p0, n];)
```

$$f[x] = 1 - 10x + 25x^2$$

$$g[x] = x - \frac{f[x]}{f'[x]}$$

$$g[x] = x - \frac{1 - 10x + 25x^2}{-10 + 50x}$$

$$g[x] = \frac{1}{10}(1 + 5x)$$

p_0 = 1.0000000000000000, $f[p_0]$ = 16.

p_1 = 0.6000000000000000, $f[p_1]$ = 4.

p_2 = 0.4000000000000000, $f[p_2]$ = 1.

p_3 = 0.2999999999999999, $f[p_3]$ = 0.2499999999999996

p_{4} = 0.2500000000000000, $f[p_{4}]$ = 0.0625

p_{5} = 0.2250000000000000, $f[p_{5}]$ = 0.015625

p_{6} = 0.2125000000000000, $f[p_{6}]$ = 0.003906249999999778

p_{7} = 0.2062500000000004, $f[p_{7}]$ = 0.0009765625

p_{8} = 0.2031250000000005, $f[p_{8}]$ = 0.000244140625

p_{9} = 0.2015625000000008, $f[p_{9}]$ = 0.00006103515625

p_{10} = 0.2007812500000012, $f[p_{10}]$ = 0.0000152587890625

p_{11} = 0.2003906250000017, $f[p_{11}] = 3.814697265625\times 10^{-6}$

p_{12} = 0.2001953125000026, $f[p_{12}] = 9.5367431640625\times 10^{-7}$

p_{13} = 0.2000976562500039, $f[p_{13}] = 2.384185793236071\times 10^{-7}$

p_{14} = 0.2000488281249604, $f[p_{14}] = 5.960464477539062\times 10^{-8}$

p_{15} = 0.2000244140624406, $f[p_{15}] = 1.490116119384766\times 10^{-8}$

p_{16} = 0.2000122070311609, $f[p_{16}] = 3.725290298461914\times 10^{-9}$

p_{17} = 0.2000061035154914, $f[p_{17}] = 9.31322352570874\times 10^{-10}$

p_{18} = 0.2000030517583396, $f[p_{18}] = 2.328306436538696\times 10^{-10}$

p_{19} = 0.2000015258796970, $f[p_{19}] = 5.820766091346741\times 10^{-11}$

p_{20} = 0.2000007629406392, $f[p_{20}] = 1.455191522836685\times 10^{-11}$

p_{21} = 0.2000003814715056, $f[p_{21}] = 3.637978807091713\times 10^{-12}$

p_{22} = 0.2000001907375319, $f[p_{22}] = 9.09494701772928\times 10^{-13}$

p_{23} = 0.2000000953714345, $f[p_{23}] = 2.273736754432321\times 10^{-13}$

p_{24} = 0.2000000476897200, $f[p_{24}] = 5.684341886080801\times 10^{-14}$

p_{25} = 0.2000000238508638, $f[p_{25}] = 1.4210854715202\times 10^{-14}$

$f[x] = 1 - 10x + 25x^2$

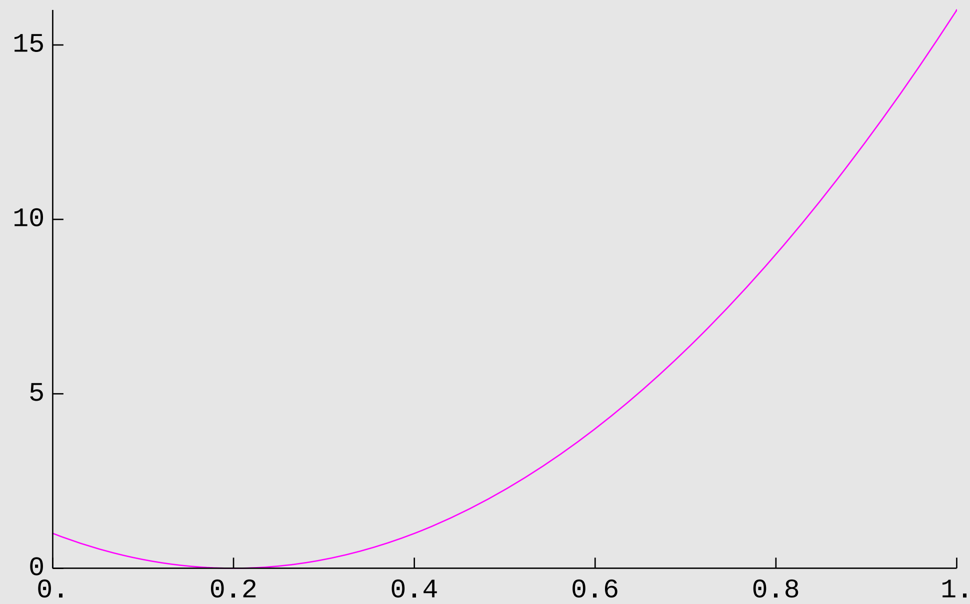
p = 0.2000000238508638

$\triangle p = \pm 2.38389\times 10^{-8}$

$f[p] = 1.4210854715202\times 10^{-14}$

$Null^4$

```
Needs["Graphics`Colors`"]
Plot[f[x], {x, 0.`, 1.`}, PlotRange → {{0, 1}, {0, 16}},
 Ticks → {Range[0, 1, 0.2`], Range[0, 16, 5]}, PlotStyle → Magenta]
```



Example 5. Convergence, Oscillation Find the solution to ArcTan[x] = 0. Use the starting approximation $p_0 = 1.35$.

Solution

```
f[x_] = ArcTan[x]; (p0 = 1.35`;) (n = 7;) (NewtonRaphson[p0, n];)
```

$f[x] = \text{ArcTan}[x]$

$g[x] = x - \frac{f[x]}{f'[x]}$

$g[x] = x - (1 + x^2)\ \text{ArcTan}[x]$

$g[x] = x - (1 + x^2)\ \text{ArcTan}[x]$

$p_0 = 1.3500000000000000, \quad f[p_0] = 0.933247528656204$

$p_1 = -1.2840911496321360, \quad f[p_1] = -0.909140878957672$

$p_2 = 1.1241231064736950, \quad f[p_2] = 0.843766774871951$

$p_3 = -0.7858718810026537, \quad f[p_3] = -0.6660666705392087$

$p_4 = 0.2915539773939717, \quad f[p_4] = 0.2836902516290485$

$p_5 = -0.0162510014432702, \quad f[p_5] = -0.01624957106691501$

$p_6 = 2.8610548941897240 \times 10^{-6}, \quad f[p_6] = 2.861054894181918 \times 10^{-6}$

$p_7 = -1.5612934800264890 \times 10^{-17}, \quad f[p_7] = -1.561293480026489 \times 10^{-17}$

$f[x] = \text{ArcTan}[x]$

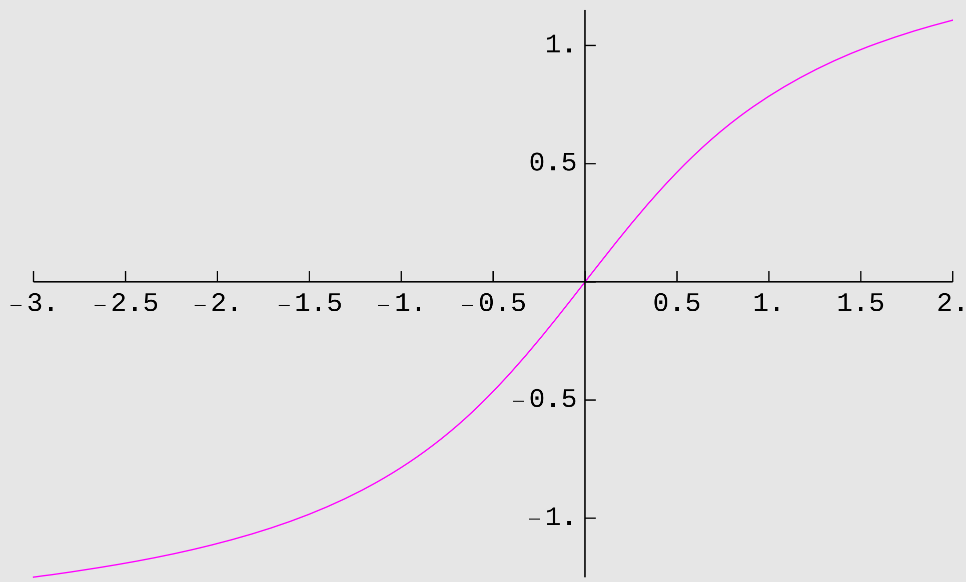
$p = -1.561293480026489 \times 10^{-17}$

$\triangle p = \pm 2.86105 \times 10^{-6}$

$f[p] = -1.561293480026489 \times 10^{-17}$

Null^3

```
Needs["Graphics`Colors`"]
Plot[f[x], {x, -3.`, 2.`}, PlotRange → {{-3.`, 2.`}, {-1.25`, 1.15`}},
 Ticks → {Range[-3, 2, 0.5`], Range[-1, 1, 0.5`]}, PlotStyle → Magenta]
```



Example 6. NON Convergence, Cycling Find the solution to $x^3 - x + 3 = 0$. Use the starting approximation $p_0 = 0.0$.

Solution

```
(f[x_] = x^3 - x + 3;) (p0 = 0.`;) (n = 16;) (NewtonRaphson[p0, n];)
```

$f[x] = 3 - x + x^3$

$g[x] = x - \frac{f[x]}{f'[x]}$

$g[x] = x - \frac{3 - x + x^3}{-1 + 3x^2}$

$g[x] = \frac{3 - 2x^3}{1 - 3x^2}$

$p_0 = 0.0000000000000000,\quad f[p_0] = 3.$

$p_1 = 3.0000000000000000,\quad f[p_1] = 27.$

$p_2 = 1.9615384615384610,\quad f[p_2] = 8.58574192080109$

$p_3 = 1.1471759614035470,\quad f[p_3] = 3.362522157362049$

$p_4 = 0.0065793714807121,\quad f[p_4] = 2.99342091332797$

p_5 = 3.0003890740712320, $f[p_5]$ = 27.01011728831863

p_6 = 1.9618181756663240, $f[p_6]$ = 8.58869137914838

p_7 = 1.1474302284816020, $f[p_7]$ = 3.363271968902603

p_8 = 0.0072562475524216, $f[p_8]$ = 2.992744134511713

p_9 = 3.0004731887732160, $f[p_9]$ = 27.0123049233781

p_{10} = 1.9618786463602410, $f[p_{10}]$ = 8.58932913619802

p_{11} = 1.1474851932167660, $f[p_{11}]$ = 3.363434113639871

p_{12} = 0.0074025013328707, $f[p_{12}]$ = 2.992597904302187

p_{13} = 3.0004924429169550, $f[p_{13}]$ = 27.01280569846047

p_{14} = 1.9618924882463070, $f[p_{14}]$ = 8.58947512636186

p_{15} = 1.1474977745445400, $f[p_{15}]$ = 3.363471231200353

p_{16} = 0.0074359752567048, $f[p_{16}]$ = 2.992564435906089

$f[x] = 3 - x + x^3$

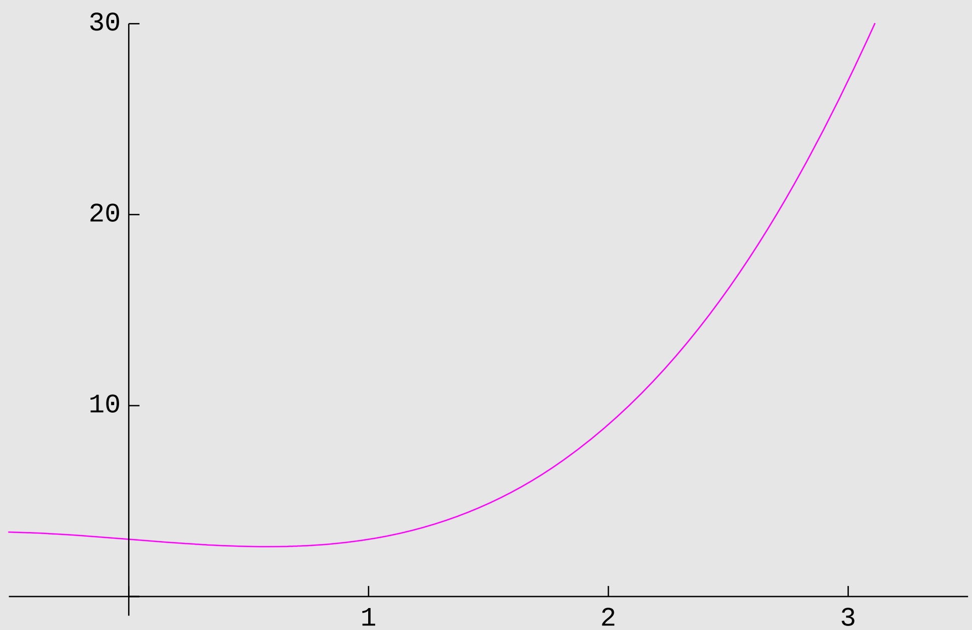
p = 0.007435975256704808

$\triangle p = \pm 1.14006$

$f[p]$ = 2.992564435906089

Null^4

```
Needs["Graphics`Colors`"]
Plot[f[x], {x, -0.5`, 3.5`}, PlotRange → {{-0.5`, 3.5`}, {-1, 30}},
 Ticks → {Range[0, 3, 1], Range[0, 30, 10]}, PlotStyle → Magenta]
```



Example 7. NON Convergence, Diverging to Infinity Find the solution to $x\, e^{-x} = 0$. Use the starting approximation $p_0 = 2.0$.

Solution

```
(f[x_] = x e^-x;) (p0 = 2.`;) (n = 16;) (NewtonRaphson[p0, n];)
```

$f[x] = e^{-x} x$

$g[x] = x - \frac{f[x]}{f'[x]}$

$g[x] = x - \frac{e^{-x} x}{e^{-x} - e^{-x} x}$

$g[x] = \frac{x^2}{-1 + x}$

$p_0 = 2.0000000000000000, \quad f[p_0] = 0.2706705664732254$

$p_1 = 4.0000000000000000, \quad f[p_1] = 0.07326255555493671$

$p_2 = 5.3333333333333330, \quad f[p_2] = 0.02574906663376769$

$p_3 = 6.5641025641025640, \quad f[p_3] = 0.00925596782007352$

p_4 = 7.7438260664067120, $f[p_4]$ = 0.003356252760816831

p_5 = 8.8921098433239900, $f[p_5]$ = 0.001222392232044362

p_6 = 10.0188186727564600, $f[p_6]$ = 0.0004463739604246538

p_7 = 11.1296979393527300, $f[p_7]$ = 0.0001632739792765629

p_8 = 12.2284175661359800, $f[p_8]$ = 0.00005979102365781943

p_9 = 13.3174773106734400, $f[p_9]$ = 0.00002191366694024704

p_{10} = 14.3986627656800000, $f[p_{10}]$ = $8.03641534421722 \times 10^{-6}$

p_{11} = 15.4732970793789400, $f[p_{11}]$ = $2.948596362407105 \times 10^{-6}$

p_{12} = 16.5423898365106000, $f[p_{12}]$ = $1.082255066644847 \times 10^{-6}$

p_{13} = 17.6067300062347600, $f[p_{13}]$ = $3.973498180122668 \times 10^{-7}$

p_{14} = 18.6669465569719900, $f[p_{14}]$ = $1.459222106648405 \times 10^{-7}$

p_{15} = 19.7235494338061500, $f[p_{15}]$ = $5.35989633653227 \times 10^{-8}$

p_{16} = 20.7769581105923200, $f[p_{16}]$ = $1.969081564010381 \times 10^{-8}$

$f[x] = e^{-x}\, x$

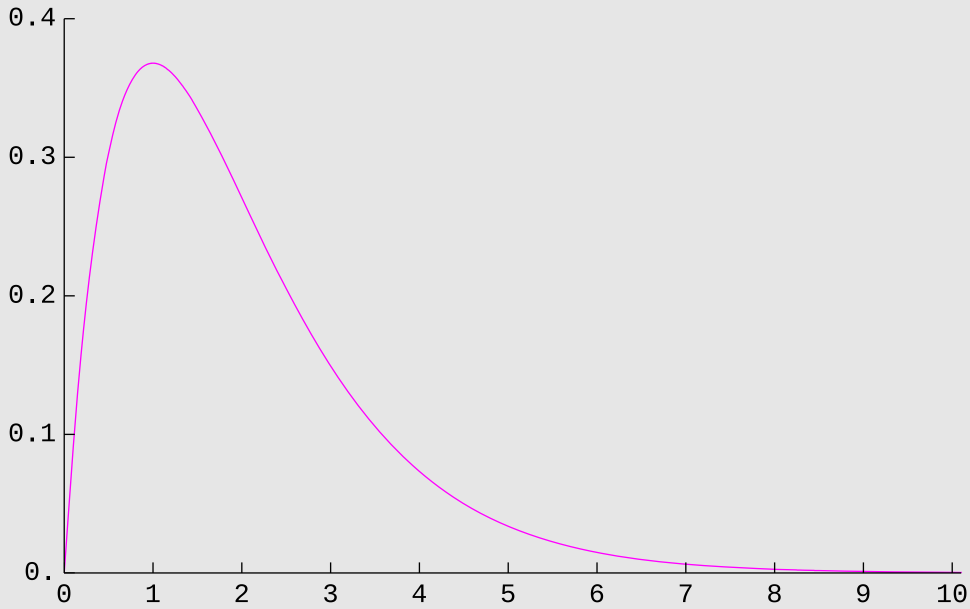
p = 20.77695811059232

$\triangle p$ = ± 1.05341

$f[p]$ = $1.969081564010381 \times 10^{-8}$

```
Null^4
```

```
Needs["Graphics`Colors`"]
Plot[f[x], {x, 0, 10.1`}, PlotRange → {{0, 10.1`}, {0, 0.4`}},
 Ticks → {Range[0, 10, 1], Range[0, 0.5`, 0.1`]}, PlotStyle → Magenta]
```



Example 8. NON Convergence, Divergent Oscillation Find the solution to ArcTan[x] = 0. Use the starting approximation $p_0 = 1.4$.

Solution

```
f[x_] = ArcTan[x];
p0 = 1.4;
n = 10;
NewtonRaphson[p0, n];
```

f[x] = ArcTan[x]

$g[x] = x - \frac{f[x]}{f'[x]}$

$g[x] = x - (1 + x^2)\ \text{ArcTan}[x]$

$g[x] = x - (1 + x^2)\ \text{ArcTan}[x]$

$p_0 = 1.4000000000000000, \quad f[p_0] = 0.950546840812075$

$p_1 = -1.4136186488037420, \quad f[p_1] = -0.955118257974891$

$p_2 = 1.4501293146283380, \quad f[p_2] = 0.967088671661225$

$p_3 = -1.5506259756377550, \quad f[p_3] = -0.99801410666222$

$p_4 = 1.8470540841501940, \quad f[p_4] = 1.074577711486838$

$p_5 = -2.8935623931424270, \quad f[p_5] = -1.238051375384995$

$p_6 = 8.7103258469833100, \quad f[p_6] = 1.456490507968295$

$p_7 = -103.2497737719132000, \quad f[p_7] = -1.561111378371298$

$p_8 = 16540.5638272396000000, \quad f[p_8] = 1.570735869363746$

$p_9 = -4.2972148289646090 \times 10^8, \quad f[p_9] = -1.570796324467808$

$p_{10} = 2.9006411728125340 \times 10^{17}, \quad f[p_{10}] = 1.570796326794897$

f[x] = ArcTan[x]

$p = 2.900641172812534 \times 10^{17}$

$\triangle p = \pm 2.90064 \times 10^{17}$

$f[p] = 1.570796326794897$

```
Needs["Graphics`Colors`"]
Plot[f[x], {x, -3.0, 2.0}, PlotRange → {{-3.0, 2.0}, {-1.25, 1.15}},
 Ticks → {Range[-3, 2, 0.5], Range[-1, 1, 0.5]}, PlotStyle → Magenta]
```